into it, hoping to stumble on a good material, gets you nowhere. But once a small number of potential candidates have been identified by the screening–ranking steps, detailed documentation can be sought for these few alone, and the task becomes viable.

Documentation for materials for the helmet visor

At this point it helps to know how the three top-ranked candidates listed in the last examples box are used. A quick web search reveals the following.

Polycarbonate
Safety shields and goggles; lenses; light fittings; safety helmets; laminated sheet for bullet-proof glazing.

Cellulose Acetate
Spectacle frames; lenses; goggles; tool handles; covers for television screens; decorative trim, steering wheels for cars.

PMMA, Plexiglas
Lenses of all types; cockpit canopies and aircraft windows; containers; tool handles; safety spectacles; lighting, automotive taillights.

This is encouraging: All three materials have a history of use for goggles and protective screening. The one that ranked highest in our list—polycarbonate—has a history of use for protective helmets. We select this material, confident that with its high fracture toughness it is the best choice.

Local conditions

The final choice between competing candidates will often depend on local conditions: in-house expertise or equipment, the availability of local suppliers, and so forth. A systematic procedure cannot help here; the decision must instead be based on local knowledge. This does not mean that the result of the systematic procedure is irrelevant. It is always important to know which material is best, even if for local reasons you decide not to use it.

We explore documentation more fully later. Here we focus on the derivation of property limits and indices.

5.3 MATERIAL INDICES

Constraints set property limits. Objectives define material indices, for which we seek extreme values. When the objective is not coupled to a constraint, the material index is a simple material property. When, instead, the two [better] are coupled, the index becomes a group of properties like those cited above. Where do they come from? This section explains.

Think for a moment of the simplest of mechanical components. The loading on a component can generally be decomposed into some combination of axial tension, bending, torsion, and compression. Almost always, one mode dominates. So common is this that the functional name given to the component describes the way it is loaded: *Ties* carry tensile loads; *beams* and *panels* carry bending moments; *shafts* carry torques; *columns* carry compressive axial loads. The words "tie," "beam," "shaft," and "column" each imply a function. Here we explore constraints, objectives, and resulting material indices for some of these.

The life energy and emissions for transport systems are dominated by the fuel consumed during use. The lighter the system is made, the less fuel it consumes and the less carbon it emits. So a good starting point is *minimum weight design*, subject, of course, to the other necessary constraints, of which the most important here have to do with stiffness and strength. We consider the generic components shown in Figure 5.6: a tie, a panel, and beams, loaded as shown.

Minimizing mass: A light, strong tie A design calls for a tie like those of the biplane in the cover picture. It must carry a tensile force F^* without failure and be as light as possible (Figure 5.6(a)). The length L is specified but the cross-section area A is not. Here, "maximizing performance" means "minimizing the mass while still carrying the load F^* safely." The design requirements, translated, are listed in Table 5.2.

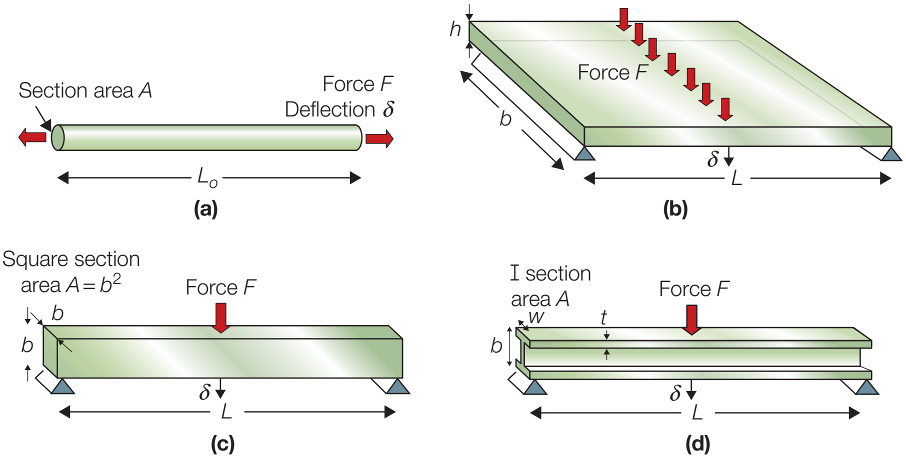

FIGURE 5.6

Generic components: (a) a tie, a tensile component; (b) a panel, loaded in bending; (c) and (d) beams, loaded in bending.

Table 5.2 Design Requirements for the Light, Strong Tie

Function	Tie rod
Constraints	Length L is specified (geometric constraint) Tie must support axial tensile load F^* without failing (functional constraint)
Objective	Minimize the mass m of the tie
Free variables	Cross-section area A Choice of material

We first seek an equation describing the quantity to be maximized or minimized. Here it is the mass m of the tie and it is a minimum that we seek. This equation, called *the objective function*, is

$$m = A\,L\,\rho \tag{5.1}$$

where A is the area of the cross-section and ρ is the density of the material of which it is made. The length L and force F are specified and are therefore fixed; the cross-section A, is free. We can reduce the mass by reducing the cross-section, but there is a constraint: A must be sufficient to carry F^*, requiring that

$$\frac{F^*}{A} \le \sigma_f \tag{5.2}$$

where σ_f is the failure strength. Eliminating A between these two equations gives

$$m \ge \underbrace{(F^*)}_{\textit{Functional constraint}}\underbrace{(L)}_{\textit{Geometric constraint}}\underbrace{\left(\frac{\rho}{\sigma_f}\right)}_{\textit{Material properties}} \tag{5.3}$$

Note the form of this result. The first bracket contains the specified load F. The second bracket contains the specified geometry (length L of the tie). The last bracket contains the material properties. The lightest tie that will carry F^* safely[2] is that made of the material with the smallest value of ρ/σ_f. We could define this as the material index of the problem, seeking a minimum, but it is more usual when dealing with specific properties to express them in a form for which a maximum is sought. We therefore invert the material

[2] In reality a safety factor, S_f, is always included in such a calculation, so that Equation (5.2) becomes $F/A = \sigma_f/S_f$. If the same safety factor is applied to each material, its value does not influence the choice. We omit it here for simplicity.

properties in Equation (5.3) and define the material index M_t (subscript t for tie) as:

$$M_{t1} = \frac{\sigma_f}{\rho} \tag{5.4}$$

The lightest tie-rod that will carry F^* without failing is that with the largest value of this index, the "specific strength," plotted in Figure 4.6. A similar calculation for a light *stiff* tie (one for which the stiffness S rather then the strength σ_f is specified) leads to the index

$$M_{t2} = \frac{E}{\rho} \tag{5.5}$$

where E is Young's modulus. This time the index is the "specific stiffness," also shown in Figure 4.6. The material group (rather than just a single property) appears as the index in both cases because minimizing the mass m—the objective—was coupled to the constraints of carrying the load F without failing or deflecting too much.

Note the procedure. The length of the rod is specified but we are free to choose the cross-section area A. The objective is to minimize its mass m. We write an equation for m: It is the objective function. But there is a constraint: The rod must carry the load F without yielding (in the first example) or bending too much (in the second). Use this to eliminate the free variable A and read off the combination of properties, M, to be maximized. It sounds easy and it is, so long as you are clear from the start what the constraints are, what you are trying to maximize or minimize, which parameters are specified, and which are free.

That was easy. Now for some slightly more difficult (and important) examples.

Minimizing Mass: A light, stiff panel A *panel* is a flat slab, like a table top. Its length L and width b are specified but its thickness is free. It is loaded in bending by a central load F (see Figure 5.6(b)). The stiffness constraint requires that it must not deflect more than δ . The objective is to achieve this with minimum mass, m. Table 5.3 summarizes the design requirements.

Table 5.3 Design Requirements for a Light, Stiff Panel

Function	Panel
Constraints	Bending stiffness S^* specified (functional constraint) Length L and width b specified (geometric constraints)
Objective	Minimize mass m of the panel
Free variables	Panel thickness h Choice of material

The objective function for the mass of the panel is the same as that for the tie:

$$m = A\,L\,\rho = b\,h\,L\,\rho$$

Its bending stiffness S must be at least S^*:

$$S = \frac{C_1 E I}{L^3} \geq S^* \tag{5.6}$$

Here C_1 is a constant that depends only on the distribution of the loads—we don't need its value (you can find it in Appendix B). The second moment of area, I, for a rectangular section is

$$I = \frac{b\,h^3}{12} \tag{5.7}$$

We can reduce the mass by reducing h, but only so far that the stiffness constraint is still met. Using the last two equations to eliminate h in the objective function gives

$$m = \underbrace{\left(\frac{12S^*}{C_1 b}\right)^{1/3}}_{\textit{Functional constraint}} \underbrace{(bL^2)}_{\textit{Geometric constraints}} \underbrace{\left(\frac{\rho}{E^{1/3}}\right)}_{\textit{Material properties}} \tag{5.8}$$

The quantities S^*, L, b, and C_1 are all specified; the only freedom of choice left is that of the material. The index is the group of material properties, which we invert such that a maximum is sought: The best materials for a light, stiff panel are those with the greatest values of

$$M_{p1} = \frac{E^{1/3}}{\rho} \tag{5.9}$$

Repeating the calculation with a constraint of strength rather than stiffness leads to the index

$$M_{p1} = \frac{\sigma_y^{1/2}}{\rho} \tag{5.10}$$

These don't look much different from the previous indices, E/ρ and σ_y/ρ, but they are: They lead to different choices of material, as we shall see in a moment.

Now for another bending problem in which shape plays a role.

Minimizing mass: A light, stiff beam Beams come in many shapes: solid rectangles, cylindrical tubes, I-beams, and more. Some of these have too many free geometric variables to apply the previous method directly. However, if we constrain the shape to be *self-similar* (such that all dimensions of the cross-section change in proportion as we vary the overall size), the problem becomes tractable again. We therefore consider beams in two stages: first, we identify the optimum materials for a light, stiff beam of a prescribed simple shape (a square section); second, we explore how much lighter it could be made for the same stiffness by using a more efficient shape.

Consider a beam of square section $A = b \times b$ that may vary in size but the square shape is retained. It is loaded in bending over a span of fixed length L with a central load F (see Figure 5.6(c)). The stiffness constraint is again that it must not deflect more than δ under F, with the objective that the beam should again be as light as possible. Table 5.4 summarizes the design requirements.

Proceeding as before, the objective function for the mass is

$$m = A\,L\,\rho = b^2\,L\,\rho \tag{5.11}$$

The bending stiffness S of the beam must be at least S^*:

$$S = \frac{C_2\,EI}{L^3} \geq S^* \tag{5.12}$$

where C_2 is a constant (Appendix B). The second moment of area, I, for a square section beam is

$$I = \frac{b^4}{12} = \frac{A^2}{12} \tag{5.13}$$

For a given L, S^* is adjusted by altering the size of the square section. Now eliminating b (or A) in the objective function for the mass gives

$$m = \left(\frac{12S^*L^3}{C_2}\right)^{1/2}(L)\left(\frac{\rho}{E^{1/2}}\right) \tag{5.14}$$

Table 5.4 Design Requirements for a Light, Stiff Beam

Function	Beam
Constraints	Length L is specified (geometric constraint) Section shape square (geometric constraint) Beam must support bending load F without deflecting too much, meaning that bending stiffness S is specified as S^* (functional constraint)
Objective	Minimize mass m of the beam
Free variables	Cross-section area A Choice of material

The quantities S^*, L, and C_2 are all specified or constant; the best materials for a light, stiff beam are those with the largest values of index M_b, where

$$M_{b_1} = \frac{E^{1/2}}{\rho} \tag{5.15}$$

Repeating the calculation with a constraint of strength rather than stiffness leads to the index

$$M_{b_2} = \frac{\sigma_y^{2/3}}{\rho} \tag{5.16}$$

This analysis was for a square beam, but the result in fact holds for any shape so long as the shape is held constant. This is a consequence of Equation (5.13); for a given shape, the second moment of area I can always be expressed as a constant times A^2, so changing the shape just changes the constant C_2 in Equation (5.14), not the resulting index.

As noted above, real beams have section shapes that improve their efficiency in bending, requiring less material to get the same stiffness. By shaping the cross-section it is possible to increase I without changing A. This is achieved by locating the material of the beam as far from the neutral axis as possible, as in thin-walled tubes or I-beams (see Figure 5.6 (d)). Some materials are more amenable than others to being made into efficient shapes. Comparing materials on the basis of the index in M_b therefore requires some caution; materials with lower index values may "catch up" by being made into more efficient shapes. We examine this in more detail in Chapter 9.

Minimizing material cost: Cheap ties, panels, and beams When the objective is to minimize cost rather than mass, the indices change again. If the material price is C_m \$/kg, the cost of the material to make a component of mass m is just mC_m. The objective function for the material cost C of the tie, panel or beam then becomes

$$C = m\,C_m = A\,L\,C_m\,\rho \tag{5.17}$$

Proceeding as before leads to indices that have the form of Equations (5.4), (5.5), (5.9), (5.10), (5.15), and (5.16), with ρ replaced by $C_m\,\rho$. Thus the index guiding material choice for a tie of specified strength and minimum material cost is

$$M = \frac{\sigma_f}{C_m\,\rho} \tag{5.18}$$

where C_m is the material price per kg. The index for a cheap stiff panel is

$$M_{p1} = \frac{E^{1/3}}{C_m\,\rho} \tag{5.19}$$

and so forth (It must be remembered that the material cost is only part of the cost of a shaped component; there is also the manufacturing cost—the cost to shape, join, and finish it.)

Associating material indices with components The components in the cover images of this chapter are labeled with the type of loading support and with the index that guides the choice of material to make them. The biplane typifies lightweight design, meaning that its materials are chosen to carry the design loads at minimum mass. The airport structure uses very large quantities of materials: Here the objective is to carry the design loads safely while minimizing the cost of the material. The guiding indices for each structure are derived from a single objective: minimizing mass in one case, minimizing material cost in the other. Often a design involves more than one objective: in choosing materials for the frame of a bicycle you might wish to minimize both the weight *and* the cost. That requires trade-off methods, the subject of Chapter 7.

How general are material indices?

This is a good moment to describe the method in more general terms. *Structural elements* are components that perform a physical function: They carry loads, transmit heat, store energy and so on. In short, they satisfy *functional requirements*. We have already identified examples: A tie must carry a specified tensile load; a spring must provide a given restoring force or store a given energy; a heat exchanger must transmit heat with a given heat flux, and so on.

The performance of a structural element is determined by three things: the functional requirements, the geometry, and the properties of the material of which it is made.[3] The performance P of the element is described by an equation of the form

$$P = \left[\begin{pmatrix}\text{Functional}\\ \text{requirements}, F\end{pmatrix}, \begin{pmatrix}\text{Geometric}\\ \text{parameters}, G\end{pmatrix}, \begin{pmatrix}\text{Material}\\ \text{properties}, M\end{pmatrix}\right]$$

or

$$P = f(F, G, M) \tag{5.20}$$

[3] In Chapter 9 we introduce a fourth: section shape.

where P, the *performance metric*, describes some aspect of the performance of the component: its mass, volume, cost, or life, for example; and f means "*a function of*." *Optimum design* is the selection of the material and geometry that maximize or minimize P, according to its desirability or otherwise.

The three groups of parameters in Equation (5.20) are said to be *separable* when the equation can be written

$$P = f_1(F) \cdot f_2(G) \cdot f_3(M) \tag{5.21}$$

where f_1, f_2, and f_3 are separate functions that are simply multiplied together. It turns out that, commonly, they are. When this is so the optimum choice of material becomes independent of the details of the design; it is the same for *all* geometries, G, and for all values of the function requirement, F. Then the optimum subset of materials can be identified without solving the complete design problem, or even knowing all the details of F and G. This enables enormous simplification: The performance for *all* F and G is maximized by maximizing f_3 (M), which is called the material efficiency coefficient, or *material index* for short. The remaining bit, f_1 $(F) \cdot f_2$ (G), is related to *the structural efficiency coefficient*, or *structural index*. We don't need it now, but we will examine it briefly in Section 5.6.

Each combination of function, objective, and constraint leads to a material index (Figure 5.7); the index is characteristic of the combination and thus of the function the component performs. The method is general and, in later

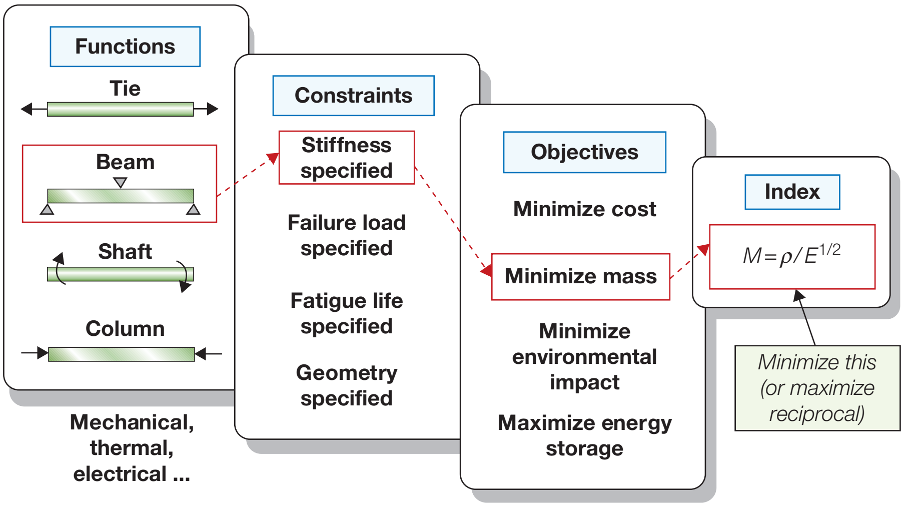

FIGURE 5.7

The specification of function, objective, and constraint leads to a materials index. The combination in the highlighted boxes leads to the index $E^{1/2}/\rho$.

Table 5.5 Examples of Material Indices

Function, Objective, and Constraints	Index
Tie, minimum weight, stiffness prescribed	$\frac{E}{\rho}$
Beam, minimum weight, stiffness prescribed	$\frac{E^{1/2}}{\rho}$
Beam, minimum weight, strength prescribed	$\frac{\sigma_y^{2/3}}{\rho}$
Beam, minimum cost, stiffness prescribed	$\frac{E^{1/2}}{C_m \rho}$
Beam, minimum cost, strength prescribed	$\frac{\sigma_y^{2/3}}{C_m \rho}$
Column, minimum cost, buckling load prescribed	$\frac{E^{1/2}}{C_m \rho}$
Spring, minimum weight for given energy storage	$\frac{\sigma_y^2}{E \rho}$
Thermal insulation, minimum cost, heat flux prescribed	$\frac{1}{\lambda C_p \rho}$
Electromagnet, maximum field, temperature rise prescribed	$\frac{C_p \rho}{\rho_e}$

ρ = density; E = Young's modulus; σ_y = elastic limit; C_m = cost/kg; λ = thermal conductivity; ρ_e = electrical resistivity; C_p = specific heat

chapters, is applied to a wide range of problems. Table 5.5 gives examples of indices and the design problems that they characterize. A fuller catalog of indices is given in Appendix C. New problems throw up new indices, as the case studies of the next chapter will show.

5.4 THE SELECTION PROCEDURE

We can now assemble the four steps into a systematic procedure.

Translation and deriving the index

Table 5.6 lists the steps. Simplified: Identify the material attributes that are constrained by the design, decide what you will use as a criterion of excellence (to be minimized or maximized), substitute for any free variables using one of the constraints, and read off the combination of material properties that optimize the criterion of excellence.

Screening: Applying attribute limits

Any design imposes certain non-negotiable demands ("constraints") on the material of which it is made. We have explained how these are translated